Gibbs Phenomenon in Structural Mechanics

S. S. K. Tadikonda* and H. Baruh†
Rutgers University, New Brunswick, New Jersey 08903

This paper is concerned with the existence of the Gibbs phenomenon in approximate solutions to structural mechanics problems by series expansions. The concepts of complementary boundary condition and interior condition are discussed. It is demonstrated that solutions that do not consider these conditions result in the Gibbs effect and very slow convergence. Guidelines are developed to eliminate the Gibbs effect when selecting trial functions for the series expansion.

Introduction

THE most common occurrence of the Gibbs effect is in the Fourier series expansion of a function that has discontinuities in its domain, or at its boundaries.[1-3] The convergence rate of a Fourier series expansion slows down to order $\mathcal{O}(1)$ in the proximity of discontinuities.[1] Developments can be found in the literature to accelerate convergence of the series expansion.[1-3]

In structural mechanics, slow convergence or convergence to an erroneous solution has been observed in several problems. One example is the analysis of complex structures by means of substructures.[4-8] If the substructures are modeled using their individual eigensolutions as expansion functions, extremely slow convergence results.[4-7] The physical explanation is that the substructure modes, which describe free-free systems, cannot account for the nonzero shear and bending moments at the interfaces. To alleviate this problem, a number of methods have been proposed, including the use of constraint modes, branch modes, attachment modes, and acceleration techniques.[4-7] In essence, these approaches provide additional terms to the series expansions that ensure shear and moment compatibility. When the substructures are modeled using admissible functions, such as finite elements, convergence is faster, because the trial functions permit shear and moment continuity at the interfaces.[8]

Convergence is also slow in systems acted upon by point inputs.[9] The presence of a concentrated linear (torsional) spring creates a discontinuity in the internal force (moment) profile, of magnitude the spring constant times the deformation, and slows convergence when the original eigenfunctions are used as trial functions. It is demonstrated in Ref. 9 that, by addition of force modes to the system representation, the rate of convergence can be increased dramatically. The use of force modes in Ref. 9 is very similar to the use of constraint modes in substructure analysis and is an extension of methods used to accelerate convergence of Fourier series.

Slow convergence is also encountered when the trial functions used cannot satisfy the dynamic boundary conditions.[10,11] or violate the force and moment balances at the boundaries when essential boundary conditions are present.[10] A typical example is to solve for the axial deformation of a bar fixed at one end by trial functions that have a zero displacement and a zero slope at that end. Because the force balance depends on the slope of the deformation, it cannot be realized with that particular set of trial functions.

In all the preceding examples, the trial functions are legitimate admissible functions, satisfying the essential boundary conditions and the differentiability condition. The question then arises as to why convergence is slow in several methods of solution, why force and moment balances need to be considered to explain the slow convergence, and whether such problems have anything in common. In this paper, we show that the slow convergence problems described above are results of the Gibbs effect in the expansion of the solution. The Gibbs effect is present because, either due to the loading or due to the choice of trial functions, a discontinuous function is expanded by a set of continuous functions.

To demonstrate the Gibbs effect and its consequences, we consider two quantities. The first is complementary boundary conditions (CBC)[10] and the second is interior conditions (IC). Complementary boundary conditions were introduced in Ref. 10; they describe quantities that cannot be specified at the boundaries. Interior conditions describe jump discontinuities in the internal force and moment profiles. If the trial functions violate the CBC or if interior conditions are present, the Gibbs effect is encountered, and slow or noncovergence results.

Complementary Boundary Conditions and Interior Conditions

Consider the bending of a fixed-free beam of length L, as shown in Fig. 1. Attached to the beam is a linear spring at the free end of constant k_1, another linear spring of constant k_2 at $x = a$, and a torsional spring with constant k_3 at $x = b$. The kinetic energy has the form

$$K(t) = \frac{1}{2}\int_0^L m(x)\dot{u}^2(x,t)\,\mathrm{d}x \tag{1}$$

where $m(x)$ is the mass distribution and $u(x,t)$ is the spatial deformation at point x at time t. The potential energy is

$$V(t) = \frac{1}{2}\left\{\int_0^L EI(x)[u''(x,t)]^2\,\mathrm{d}x + k_1u^2(L,t) + k_2u^2(a,t) + k_3[u'(b,t)]^2\right\} \tag{2}$$

where $EI(x)$ is the stiffness and primes denote spatial differentiation. Using the extended Hamilton principle and perform-

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*Graduate Assistant, Department of Mechanical and Aerospace Engineering; currently Project Engineer, Dynacs Engineering Co., Palm Harbor, FA. Member AIAA.

†Associate Professor, Department of Mechanical and Aerospace Engineering. Member AIAA.

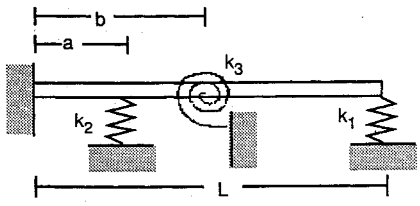

Fig. 1 Beam with discrete springs.

ing the required integrations by parts, we obtain

$$-\int_{t_1}^{t_2}\int_0^L\Big\{m(x)\ddot{u}(x,t)+\frac{\partial^2[EI(x)\partial^2u(x,t)/\partial x^2]}{\partial x^2}+k_2u(x,t)\delta(x-a)-\frac{\partial}{\partial x}[k_3u'(x,t)\delta(x-b)]\Big\}\times\delta u(x,t)\,\mathrm{d}x\,\mathrm{d}t$$
$$+\int_{t_1}^{t_2}\left[\left\{\frac{\partial[EI(x)\partial^2u(x,t)/\partial x^2]}{\partial x}-k_1u(x,t)\delta(x-L)\right\}\delta u\right]\Bigg|_0^L\mathrm{d}t-\int_{t_1}^{t_2}\left[\frac{EI(x)\partial^2u(x,t)}{\partial x^2}\delta u'\right]\Bigg|_0^L\mathrm{d}t=0 \quad (3)$$

where we have used the same symbol for the Dirac delta and variation functions simultaneously. By virtue of the arbitrariness of the variation of u, each of the terms in the above equation must vanish individually. The integrand in the first term in Eq. (3) is recognized as the equation of motion and the second and third terms as the boundary expressions. The equation of motion is

$$m(x)\ddot{u}(x,t)+\frac{\partial^2[EI(x)\partial^2u(x,t)/\partial x^2]}{\partial x^2}=k_2u(x,t)\delta(x-a)+\frac{\partial}{\partial x}[k_3u'(x,t)\delta(x-b)] \qquad 0<x<L \quad (4)$$

The boundary conditions are ascertained by examining the system geometry. Each boundary expression contains the product of two terms. One of the terms corresponds to an internal force (or moment) and the other to a displacement (or slope). For example, at the fixed end $x=0$, $u(0,t)=0$, and $u'(0,t)=0$. But the values of the internal force $\partial[EI(x)\partial^2u(x,t)/\partial x^2]/\partial x$ and internal moment $EI(x)\partial^2u(x,t)/\partial x^2$ at $x=0$ are not shown. This lack of knowledge is expected, because it is a fundamental principle of mechanics that, at a point constrained not to move, one cannot know the value of the loading until the system differential equation is solved.[13-14]

Now, set all spring constants to zero. At the free end, the geometry indicates that the displacement and slope are unknown, so that their variation is arbitrary. It follows that one must have $\partial(EI(x)\partial^2u(x,t)/\partial x^2)\partial x=0$ and $EI(x)\partial^2u(x,t)/\partial x^2=0$ at $x=L$, which are the boundary conditions at that end.

This analysis is standard and straightforward. Our purpose in repeating it here is to underscore the existence of complementary boundary conditions (CBC), defined as follows[10]: *Complementary boundary conditions are the terms in the boundary expressions that cannot be specified or determined before the boundary-value problem is solved.* It is shown in Ref. 10 that CBC must exist in all boundary-value problems that can be cast into variational form, and that there corresponds a CBC for each boundary condition.

The relationship between boundary conditions and CBCs is analogous to that between constraints and constraint forces. If we consider the boundary conditions as constraints on an otherwise free structure, then the CBCs become the constraint forces. Table 1 shows the boundary conditions and the corresponding complementary boundary conditions for a beam for various end conditions.

Table 1 Boundary conditions and CBC for beam problems

End	Boundary condition	CBC
Fixed	$u=0$	$\frac{\partial}{\partial x}\left[EI(x)\frac{\partial^2u}{\partial x^2}\right]\neq 0$
	$u'=0$	$EI(x)\frac{\partial^2u}{\partial x^2}\neq 0$
Free	$\frac{\partial}{\partial x}\left[EI(x)\frac{\partial^2u}{\partial x^2}\right]=0$	$u\neq 0$
	$EI(x)\frac{\partial^2u}{\partial x^2}=0$	$u'\neq 0$
Pinned	$u=0$	$\frac{\partial}{\partial x}\left[EI(x)\frac{\partial^2u}{\partial x^2}\right]\neq 0$
	$EI(x)\frac{\partial^2u}{\partial x^2}=0$	$u'\neq 0$

One may question the need to define a quantity called complementary boundary condition. After all, its existence is obvious, especially in structural mechanics, and it merely describes quantities whose values are not known. Our motivation to define the CBC stems from the observation that, while constructing approximate solutions to boundary-value problems, trial functions may be used that are incompatible with the system force and moment balances.[10] The incompatibility arises from violation of CBC.

We next consider the discrete spring of constant k_2. The equation of motion becomes

$$m(x)\ddot{u}(x,t)+\frac{\partial^2[EI(x)\partial^2u(x,t)/\partial x^2]}{\partial x^2}=-k_2u(x,t)\delta(x-a) \quad (5)$$

One can evaluate Eq. (5) at every point along the domain except at $x=a$, the location of the spring. In the neighborhood of a, we integrate Eq. (5) between $a-\epsilon$ and $a+\epsilon$, where ϵ is a small quantity. Taking the limit as $\epsilon\to 0$, we obtain

$$\lim_{\epsilon\to 0}\int_{a-\epsilon}^{a+\epsilon}\left\{\left[m(x)\ddot{u}(x,t)+\frac{\partial^2[EI(x)\partial^2u(x,t)/\partial x^2]}{\partial x^2}+k_2u(x,t)\delta(x-a)\right]\right\}\mathrm{d}x=\frac{\partial[EI(x)\partial^2u(x,t)/\partial x^2]}{\partial x}\Bigg|_{a-}^{a^+}+k_2u(a,t)=0 \quad (6)$$

which indicates the discontinuity in the shear profile at that point.

Using a similar approach, in the presence of the torsional spring, a relationship for the internal moment balance at $x=b$ can be obtained as

$$\frac{EI(x)\partial^2u(x,t)}{\partial x^2}\Bigg|_{b-}^{b^+}-k_3u'(b,t)=0 \quad (7)$$

which demonstrates the jump discontinuity due to the discrete spring.

We introduce a second quantity called interior conditions (IC) to represent the jump discontinuities in the shear and moment distributions of structural systems. For the general boundary-value problem of order $2p$, the definition of IC would be discontinuities in the derivatives of order p or higher of the solution.

If a solution is sought for a boundary-value problem by trial functions that violate the CBC, discontinuities are encountered at the boundaries where the CBC are violated.[10] Similarly, when concentrated loads act on a system, the actual solution has discontinuities in the higher derivatives that may not be approximated well by the trial functions used. The question then arises as to the effect of these discontinuities on the nature of the solution obtained by approximate methods, especially with respect to the order of convergence. To investigate this, we first analyze the relationships between exact and approximate solutions to boundary-value problems, and then consider the Gibbs effect.

Consider a boundary-value problem in the form

$$\ell u(x) - \lambda m u(x) = 0 \tag{8}$$

where ℓ and m are self-adjoint operators denoting the stiffness and mass, respectively. The stiffness operator is of order $2p$. The eigensolution consists of the eigenfunctions $\phi_r(x)$ and eigenvalues $\lambda_r (r = 1,2,\ldots)$. The eigenfunctions can be normalized to satisfy $[\phi_r(x), m(x)\phi_s(x)] = \delta_{rs}$, $[\phi_r(x), \ell\phi_s(x)] = \lambda_r\delta_{rs}$ $(r,s = 1,2,\ldots)$, in which $[e,f] = \int ef\, dx$. The eigensolution also satisfies

$$\ell\phi_r(x) - \lambda_r m(x)\phi_r(x) = 0 \tag{9}$$

Consider now an approximate solution of order n in the form

$$u^{[n]}(x) = \sum_{r=1}^{n} v_r \Psi_r(x) \tag{10}$$

in which the superscript n in the square brackets denotes the order of approximation, $\Psi_r(x)$ are trial functions, and v_r are undetermined coefficients.[12] Minimization of the Rayleigh's quotient leads to the algebraic eigenvalue problem of order n

$$Kv = \Lambda M v \tag{11}$$

where the entries of K and M are given by $K_{ij} = [\Psi_i(x), \Psi_j(x)]^*$, $M_{ij} = [\psi_i(x), m(x)\Psi_j(x)]$ $(i,j = 1,2,\ldots,n)$, in which the starred square brackets denote the energy inner product associated with the stiffness operator.[12] Solution of Eq. (11) yields a set of eigenvalues Λ_r and associated eigenvectors v_r, $v_r = [v_{1r} v_{2r} \ldots v_{nr}]^T$, which can be used to obtain a set of approximate eigenfunctions $\theta_r(x)$ as

$$\theta_r(x) = \sum_{i=1}^{n} v_{ir}\Psi_i(x), \qquad r = 1,2,\ldots,n \tag{12}$$

The eigenfunctions can be normalized to satisfy $[\theta_r(x), m(x)\theta_s(x)] = \delta_{rs}$, $[\theta_r(x), \theta_s(x)]^* = \Lambda_r\delta_{rs} (r,s = 1,2,\ldots,n)$. Assume for the present that the trial functions are $2p$ times differentiable. This is the case for almost all trial functions, except those in the finite element method. Considering the expansion of $u^{[n]}(x)$ in terms of $\theta_r(x)$ and the weighted residual formulation,[12] we have

$$\ell\theta_r(x) - \Lambda_r m(x)\theta_r(x) = R_r(x), \qquad r = 1,2,\ldots,n \tag{13}$$

where $R_r(x)$ is the residual associated with the rth mode, satisfying the relations $[R_r(x), \theta_s(x)] = 0$ $(r,s = 1,2,\ldots,n)$.

We expand the approximate eigenfunctions in terms of the actual eigenfunctions as

$$\theta_r(x) = \sum_{i=1}^{\infty} d_{ri}\phi_i(x), \qquad r = 1,2,\ldots,n \tag{14}$$

The coefficients d_{ri} can be found by exploiting the orthogonality properties of the eigenfunctions. Multiplication of Eq. (14) by $m(x)\phi_j(x)$ $(j = 1,2,\ldots)$, integration along the domain, and use of the orthogonality relations yield $d_{rj} = [\theta_r(x), m(x)\phi_j(x)]$ $(r = 1,2,\ldots,n; j = 1,2,\ldots)$. Next, we expand the exact eigenfunctions in terms of the approximate eigenfunctions by

$$\phi_r(x) = \sum_{i=1}^{n} c_{ri}\theta_i(x) + \Delta_r(x), \qquad r = 1,2,\ldots,n \tag{15}$$

where we once again use the orthogonality relations to find the coefficients c_{ri}. By realizing that the error $\Delta_r(x)$ is orthogonal to $\theta_i(x)$, $[\Delta_r(x), m(x)\theta_i(x)] = 0$ $(r,i = 1,2,\ldots,n)$, and multiplying Eq. (15) by $m(x)\theta_j(x)$ $(j = 1,2,\ldots,n)$, we arrive at

$$c_{rj} = [\phi_r(x), m(x)\theta_j(x)] = d_{jr}, \qquad j,r = 1,2,\ldots,n \tag{16}$$

Orthogonal Series Expansions and Gibbs Phenomenon

For a function $f(x)$ defined between 0 and 2π, its Fourier series expansion is given by the function $g(x)$ with period 2π, such that

$$g(x) = b_0 + \sum_{r=1}^{\infty}(a_r \sin rx + b_r \cos rx) \tag{17}$$

where the coefficients a_r and b_r can be found from[1-3]

$$a_r = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin rx\, dx \tag{18a}$$

$$b_r = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos rx\, dx \tag{18b}$$

It can be shown[3] that if $f(x)$ is piecewise continuous and has bounded total variation

$$g(x) = 0.5\,[f(x^+) + f(x^-)] \qquad 0 < x < 2\pi \tag{19}$$

We would like to examine the convergence of Fourier series expansions and analyze the Gibbs effect. We are especially interested in convergence in the neighborhood of a discontinuity. Denoting by $g_n(x)$ the nth order expansion, and by x_o the point of discontinuity in $f(x)$, one can show that as $n \to \infty$[3]

$$g_n(x) - \tfrac{1}{2}[f(x_o^+) + f(x_o^-)] = \mathcal{O}(1) \tag{20}$$

which implies that convergence of $g_n(x)$ to $f(x)$ is not uniform in the neighborhood of a discontinuity. This is known as the Gibbs phenomenon and results in oscillations and an overshoot in the plots of $g_n(x)$ in the neighborhood of the discontinuity. One can quantify the amount of overshoot for a given problem.[13]

Next, we examine the amplitudes of the expansion coefficients. Consider a Fourier sine series. If $f(x)$ has continuous derivatives up to order $k-1$, one can integrate Eq. (18a) k times by parts to obtain

$$\begin{aligned} a_r &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin rx\, dx = \frac{1}{\pi r}\int_0^{2\pi} f'(x)\cos rx\, dx \\ &= \frac{-1}{\pi r^2}\int_0^{2\pi} f''(x)\sin rx\, dx = \cdots \\ &= \frac{1}{\pi(-1)^{(k/2)} r^k}\int_0^{2\pi} f^{(k)}(x)\sin rx\, dx \end{aligned} \tag{21}$$

where, without loss of generality, we have assumed that $f(x)$ and its even derivatives vanish at $x = 0$ and $x = 2\pi$, and k is even. If k is odd then there will be a cosine term in the integrand. It is clear that the order of the coefficients in the Fourier series expansion depends on how many times the function $f(x)$ is differentiable. If the function $f(x)$ is $k-1$ times differentiable, and $f^{(k)}$ is integrable, then the coefficients $a_r = \mathcal{O}(1/r^k)$. Also, as $n \to \infty$, at a point away from a discontinuity in $f^{(k-1)(x)}$ we have[3]

$$g_n(x) - f(x) = \mathcal{O}(1/n^k) \tag{22}$$

and in the neighborhood of the discontinuity

$$g_n(x) - ½[f(x_o^+) + f(x_o^-)] = \mathcal{O}(1/n^{k-1}) \tag{23}$$

Several approaches have been proposed in the literature to accelerate convergence of Fourier series, e.g., Refs. 1 and 2. One approach is as follows. Consider a function $f(x)$, with a discontinuity at $x = \beta$. From the above discussion, the coefficients in the Fourier series expansion of $f(x)$ will be of order $\mathcal{O}(1/r)$. One can express $f(x)$ as

$$f(x) = f_d(x) + f_c(x) \tag{24}$$

where $f_c(x)$ is a continuous function and $f_d(x)$ is discontinuous at $x = \beta$, with $f(0) = f(2\pi) = 0$ and $f(\beta^+) - f(\beta^-) = f_d(\beta^+) - f_d(\beta^-)$. Because $f_c(x)$ is continuous, the magnitude of the coefficients a_r in its Fourier series expansion will be of order $\mathcal{O}(1/r^2)$ or lower. Hence, convergence has been accelerated. If the discontinuity is in a derivative of $f(x)$, say in $f'(x)$ and at $x = \beta$, then $f_d(x)$ is selected such that $f_d'(x)$ is discontinuous at $x = \beta$ and $f'(\beta^+) - f'(\beta^-) = f_d'(\beta^+) - f_d'(\beta^-)$.

Gibbs Effect in Eigenfunction Expansions

The results of the previous section are applicable to any orthogonal series expansion of a function. Here, we extend them to eigenfunction expansions. To this end, we recall the boundary-value problem defined in Eq. (8). When ℓ is self-adjoint, it has the general form

$$\ell = h_0(x) + \frac{d}{dx}\left[h_1(x)\frac{d}{dx}\right] + \cdots + \frac{d^p}{dx^p}\left[h_p(x)\frac{d^p}{dx^p}\right] \tag{25}$$

where $h_0(x)$, $h_1(x), \ldots, h_p(x)$ are algebraic functions. For example, for a beam, $p = 2$, and $h_0(x)$ corresponds to a continuously distributed linear spring, $h_1(x)$ to a distributed torsional spring or an axial load, and $h_2(x)$ to the beam stiffness. Consider now a function $f(x)$ defined between 0 and L and expand it in terms of the normalized eigenfunctions $\phi_r(x)$ $(r = 1,2, \ldots)$ as

$$f(x) = \sum_{r=1}^{\infty} a_r \phi_r(x) \tag{26a}$$

$$a_r = \int_0^L f(x) m(x) \phi_r(x)\, dx \tag{26b}$$

Considering Eq. (9) and substituting for $m(x)\phi_r(x)$, we obtain[3]

$$a_r = \frac{1}{\lambda_r}\int_0^L f(x)\ell\phi_r(x)\, dx, \qquad r = 1,2, \ldots \tag{27}$$

Next, we integrate Eq. (27) by parts $2p$ times, which yields

$$a_r = \frac{1}{\lambda_r}\Bigg\{f(x)h_1(x)\phi_r'(x) - f'(x)h_1(x)\phi_r(x) + f(x)\frac{d}{dx}[h_2(x)\phi_r''(x)] - f'(x)h_2(x)\phi_r''(x) + f''(x)h_2(x)\phi_r'(x) - \phi_r(x)\frac{d}{dx}[h_2(x)f''(x)] + \cdots - \phi_r(x)\frac{d^{p-1}}{dx^{p-1}}[h_p(x)f^{(2p)}(x)]\Bigg\}\Bigg|_0^L + \frac{1}{\lambda_r}\int_0^L \phi_r(x)\ell f(x)\, dx, \qquad r = 1,2, \ldots \tag{28}$$

It is shown in Ref. 3 that the last term in the preceding equation is of order $\mathcal{O}(1/\lambda_r)$ or lower. To investigate the Gibbs effect, we generalize the results in Ref. 15 with respect to the asymptotic behavior of the eigensolution of self-adjoint boundary-value problems to the following. As $r \to \infty$, the asymptotic behavior of the eigensolution is governed by

$$\lambda_r \sim Cr^{2p}, \qquad \phi_r(x) \sim f(Drx) \tag{29}$$

in which the constants C and D depend on $h_p(x)$ and $m(x)$.[15] When $p = 1$, such as in the wave equation

$$f(Drx) = A_r \sin(Drx) + B_r \cos(Drx)$$

and when $p = 2$, such as in the beam equation

$$f(Drx) = A_r \sin(Drx) B_r \cos(Drx) + A_r' \sinh(Drx) + B_r' \cosh(Drx)$$

Consider, for example, a beam in transverse vibration pinned at both ends. The boundary conditions are $\phi_r(0) = 0$, $\phi_r(L) = 0$, $h_2(0)\phi_r''(0) = 0$, $h_2(L)\phi_r''(L) = 0$. If we wish to expand a continuous function $f(x)$ that satisfies the boundary conditions $f(L) = 0$, $f''(0) = 0$, $f''(L) = 0$, but violates the fourth one by $f(0) \neq 0$, using $\phi_r(x)$ we obtain from Eq. (28)

$$a_r = \frac{1}{\lambda_r}\left\{f(0)h_1(0)\phi_r'(0) + f(x)\frac{d}{dx}[h_2(x)\phi_r''(x)]\bigg|_{x=0}\right\} + \mathcal{O}\left(\frac{1}{\lambda_r}\right) \tag{30}$$

Considering the asymptotic behavior of the eigensolution given in Eqs. (29), we have

$$a_r = \mathcal{O}\left(\frac{1}{r^4}r\right) + \mathcal{O}\left(\frac{1}{r^4}r^3\right) + \mathcal{O}\left(\frac{1}{\lambda_r}\right) = \mathcal{O}(1/r) \tag{31}$$

so that existence of the Gibbs effect is obvious.

If the function $f(x)$ has a discontinuity in $[0,L]$, then Eq. (27) cannot be integrated by parts. Referring to Eq. (28) and recalling the asymptotic behavior of the eigenfunctions, we conclude that the coefficients in the expansion must be of order $\mathcal{O}(1/r)$. Similarly, if $f(x)$ satisfies the boundary conditions but has a discontinuity in a higher order derivative, for example k, then the coefficients of the expansion will be of order $\mathcal{O}(1/r^{k+1})$.

Interior conditions exist in structures acted upon by discontinuous forces and moments. If a solution $u^{[n]}(x)$ is sought using the eigenfunctions of the system without the discontinuous forces and moments, a Gibbs effect will be encountered. For example, in beam problems, in the presence of a concentrated moment or a torsional spring, there will be a discontinuity in the second derivative of the deformation and the coefficients a_r will be of order $\mathcal{O}(1/r^3)$. In the presence of a concentrated force or a linear spring, a_r will be of order $\mathcal{O}(1/r^4)$.

One way of accelerating convergence in structures acted upon by concentrated loads is by the addition of force modes to the expansion of the solution.[9] Force modes are the solution of the static problem associated with the actual structure, or a simplified version of the actual structure, and they can account for the discontinuity in the higher order derivatives of the solution. This issue will be discussed in more detail within the context of a numerical example in the next section.

We next analyze the Gibbs effect associated with eigenfunctions obtained from an approximate analysis. The interest lies in cases in which the complementary boundary conditions are violated. The expansion of a function $f(x)$ in terms of the approximate eigenfunctions $\theta_r(x)$ is

$$f(x) = \sum_{r=1}^{n} b_r \theta_r(x) + e(x) \tag{32}$$

in which $e(x)$ is the error, and it satisfies the relation $[e(x), \theta_r(x)] = 0$ $(r = 1,2, \ldots, n)$. Using a procedure similar to that in Eq. (28), we obtain for b_r

$$b_r = \frac{1}{\Lambda_r}\Big\{f(x)h_1(x)\theta_r'(x) - f'(x)h_1(x)\theta_r(x) + f(x)\frac{d}{dx}[h_2(x)\theta_r''(x)] - f'(x)h_2(x)\theta_r''(x) + f''(x)h_2(x)\theta_r'(x) - \theta_r(x)\frac{d}{dx}[h_2(x)f''(x)] + \cdots - \theta_r(x)\frac{d^{p-1}}{dx^{p-1}}[h_2(x)f^{(p)}(x)]\Big\}\Big|_0^L + \frac{1}{\lambda_r}\int_0^L[\theta_r(x)\ell f(x) - \theta_s(x)R_r(x)]\,dx, \qquad r = 1,2,\ldots n \tag{33}$$

The additional term is due to the residual $R_r(x)$ which is defined in Eq. (13). When expanding $f(x)$ using $\theta_r(x)$, one cannot ascertain the order of the expansion coefficients. This is because the asymptotic behavior of the approximate eigensolution is not known. Even if one assumes that the asymptotic behavior is similar to that given by Eqs. (29), the order of the residual $R_r(x)$ must be known before the order of b_r can be determined. Intuitively, one can expect the convergence rate to depend on the derivative at which the Gibbs effect is encountered. In the next section, we will quantify the error for such systems using concepts from boundary-layer theory.

To illustrate the Gibbs effect, consider the case where the approximate eigenfunctions are generated using trial functions that violate the CBC. An example is the axial deformation of fixed-free bar, for which $p = 1$, $h_0(x) = 0$, and $h_1(x) = -EA(x)$. The boundary conditions are $u(0) = 0$, and $h_1(L)u'(L) = 0$. The corresponding CBC are $h_1(0)u'(0) \neq 0$, and $u(L) \neq 0$. Let us use a set of trial functions $\Psi_r(x)$ that satisfy the essential boundary condition, such that $\Psi_r(0) = 0$, but violate the CBC at $x = 0$ by having $\Psi_r'(0) = 0$ as well. The computed eigenfunctions $\theta_r(x)$ will then have zero displacement and slope at $x = 0$.

We next expand the exact eigenfunctions $\phi_s(x)$ by $\theta_r(x)$. We repeat Eq. (15)

$$\phi_s(x) = \sum_{r=1}^{n} b_{sr}\theta_r(x) + \Delta_s(x) \tag{34}$$

Differentiating this equation and evaluating it at the point where the CBC is violated ($x = 0$), we obtain

$$\Delta_s'(0) = \phi_s'(0) - \sum_{r=1}^{n} b_{sr}\theta_r'(0) = \phi_s'(0) \tag{35}$$

which implies that, in the neighborhood of $x = 0$, the derivative $\Delta_s'(x)$ is of order $\mathcal{O}(1)$, clearly indicating nonuniform convergence and the Gibbs effect [see Eq. (20)] in the first derivative.

We conclude from this that because the exact solution itself satisfies all boundary conditions and does not violate the CBC, using trial functions that violate the CBC leads to slow and possibly nonconvergence. In addition, such a choice also creates a physical or geometric incompatibility. The trial functions that violate the CBC at $x = 0$ set the internal force (the constraint force) at that point to be zero, which is physically impossible. A revised definition for both admissible and comparison functions is proposed in Ref. 10, which calls for the consideration of the CBC when selecting admissible and comparison functions.

It is natural to examine the issue of completeness of the basis when the CBC are violated or when initial conditions are present. Consider the set of n admissible functions $\Psi_1(x)$, $\Psi_2(x), \ldots, \Psi_n(x)$ in Eq. (10). Because these functions are from a complete set, they are p times differentiable and satisfy the essential boundary conditions. They constitute a complete basis for the n-dimensional energy subspace. That is, for example, for beam problems, the displacement profile $u(x)$ and slope $u'(x)$ can be expanded in a uniformly convergent series, and the following expression is valid in addition to Eq. (10):

$$u'(x) = \sum_{i=1}^{n} v_i\Psi_i'(x) \tag{36}$$

The question then arises whether the trial functions, and hence the resulting eigenfunctions, provide a basis for the shear and bending moment distributions as well. Mathematically, for beam problems, given the expansion in Eq. (10), will the expansions

$$u''(x) = \sum_{i=1}^{n} v_i\Psi_i''(x), \qquad u'''(x,t) = \sum_{i=1}^{n} v_i\Psi_i'''(x) \tag{37}$$

hold at every point x in the domain? The answer depends on the continuity and differentiability of the actual solution $u(x)$, as well as that of the trial functions $\Psi_i(x)$. If the trial functions have continuous derivatives up to order $2p$ while the actual solution has discontinuities in derivatives higher that p, then the above expansions may not be valid and hence the trial functions will not provide a complete basis for the derivatives higher than order $p - 1$.

We conclude that in order to provide a complete basis for the internal force and moment, the trial functions need to not violate the CBC as well as to satisfy the interior conditions. Otherwise, the basis is no longer complete and the Gibbs effect is observed.

Examples to the Gibbs Effect in Structural Mechanics

In this section, we will illustrate the Gibbs effect and discuss convergence issues for structural mechanics problems.

Interior Conditions

Consider the vibration of a pinned-pinned uniform beam. The beam admits a closed-form eigensolution of the form $\lambda_r = \omega_r^2 = (r\pi/L)^4 EI/m$, $\phi_r(x) = \sqrt{2/mL}\,\sin r\pi x/L$ $(r = 1,2,\ldots)$. We select all parameters to be unity, except $m = \pi^4$. First, we analyze the static case and place a concentrated force F at $x = \beta$, and obtain the deflection $u(x)$ by an expansion of the eigenfunctions of the original system, as

$$u^{[n]}(x) = \sum_{r=1}^{n} a_r\phi_r(x) \tag{38}$$

The potential energy can be shown to be

$$V = \frac{1}{2}[u^{[n]}(x), u^{[n]}(x)]^* = \frac{1}{2}\sum_{r=1}^{n} a_r^2\lambda_r \tag{39}$$

The virtual work has the form

$$\delta W = \int f(x)\delta u^{[n]}(x)\,dx = \int F\delta(x-\beta)\delta\sum_{r=1}^{n} a_r\phi_r(x)\,dx = \sum_{r=1}^{n} F\phi_r(\beta)\delta a_r \tag{40}$$

Using the principle of virtual work, we obtain n independent equations for the coefficients a_r

$$a_r = \frac{F\phi_r(\beta)}{\lambda_r} = \frac{F\sqrt{2}\,\sin r\pi\beta}{\pi^2 r^4}, \qquad r = 1,2,\ldots,n \tag{41}$$

As expected, because there is a discontinuity in the shear profile, the coefficients a_r are of order $\mathcal{O}(1/r^4)$. Note that, because of the decoupled nature of the equations, Eq. (41) is valid for all values of r.

If we replace the concentrated force by a concentrated moment of magnitude M at $x = \beta$, the virtual work expression becomes

$$\delta W = -\int M(x)\delta\left[\frac{\partial u^{[n]}(x)}{\partial x}\right] dx$$
$$= \int M\delta(x-\beta)\,\delta\sum_{r=1}^{n} a_r\phi_r'(x)\,dx = \sum_{r=1}^{n} M\phi_r'(\beta)\,\delta a_r \qquad (42)$$

and the coefficients a_r become

$$a_r = \frac{M\phi_r'(\beta)}{\lambda_r} = \frac{M\sqrt{2}\cos r\pi\beta}{\pi r^3}, \qquad r = 1,2,\ldots,n \qquad (43)$$

Again, as expected, because the interior condition is in the second derivative, the coefficients of the expansion are of order $\mathcal{O}(1/r^3)$. We conclude that for comparable accuracy more terms need to be used in the series expansion when the excitation is a concentrated moment than when the excitation is a force.

Next, we consider the dynamic case. We first place a concentrated torsional spring of stiffness S at $x = \beta$ (as shown in Fig. 2) and use Eq. (38) to find an approximate solution. The actual eigenvalues can be found by solving the associated characteristic equation.

To improve convergence, we consider a solution that includes a force mode in the form[9]

$$u^{[n]}(x) = \sum_{r=1}^{n} a_r\phi_r(x) + b\gamma(x) \qquad (44)$$

in which $\gamma(x)$ is the force mode, which is the deformation profile associated with the static problem of a pinned-pinned uniform beam with a concentrated moment at $x = \beta$.[9]

The reader is referred to Ref. 9 for further details. Introducing the notation $\Psi_1(x) = \gamma(x)$, $\Psi_r(x) = \phi_{r-1}(x)(r = 2,3,\ldots, n+1)$ and using Eq. (6), we obtain the algebraic eigenvalue problem defined in Eq. (11) of order $N = n + 1$, where the

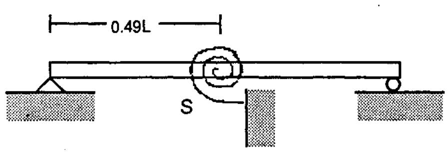

Fig. 2 Pinned-pinned beam with a torsional spring.

mass and stiffness matrices have the form

$$M_{ij} = [\Psi_i(x),\, m\Psi_j(x)], \qquad K_{ij} = [\Psi_i(x),\Psi_j(x)]^*$$
$$= \int_0^1 EI\Psi_i''(x)\Psi_j''(x)\,dx + S\Psi_i'(\beta)\Psi_j'(\beta) \qquad (45)$$

The eigenvalues obtained from the preceding analysis without and with the force mode, along with the actual values from the characteristic equation, are shown in Table 2 for $\beta = 0.49$ and $S = 500\pi^4$. Also shown in Table 2 are frequencies calculated from a pseudocharacteristic equation formulated in Ref. 16 for $n = 10$, 20, and 100. Note that we refer to the equation obtained by the method of Ref. 16 as the pseudocharacteristic equation to distinguish it from the exact characteristic equation.

The accuracy of the pseudocharacteristic equation is better than that of the Rayleigh-Ritz method without the force mode. Convergence for both approaches is slow. In contrast, addition of the force mode accelerates convergence substantially.

Another observation from the same table is that the convergence rates for the even- and odd-numbered eigenvalues are different. Convergence to the odd-numbered eigenvalues is better. This is because the second derivatives of the even-numbered eigenfunctions of a pinned-pinned beam vanish at $x = 0.5L$. The moment balance at $x = 0.49L$ is more difficult to achieve for the even-numbered eigenfunctions, resulting in slower convergence to the even-numbered eigenvalues.

We next replace the torsional spring with a linear spring of constant $S = 500\pi^4$ and conduct the same eigenvalue analysis. The results are shown in Table 3. Comparing these results with the torsional spring case, we observe that convergence is much faster for all types of approximations. This is to be expected, because the Gibbs effect is now in the third derivatives, as opposed to the second derivative when the torsional spring is present.

We also observe from Table 3 that convergence to the odd-numbered eigenvalues is better than the even-numbered modes. At first this appears surprising, because we would expect the opposite of the torsional spring case to occur here. An examination of the plots of the eigenfunctions explains the reason.[17] It turns out that the linear spring makes the first eigenfunction of the system look very similar to the second eigenfunction of the beam without the spring.

Table 2 Comparison of accuracies of various methods for the beam with a torsional spring

	Rayleigh-Ritz: without forced modes		Pseudocharacteristic equation[16]			Rayleigh-Ritz: 1 force-mode,	Exact values from charac-
Frequency	$n = 10$	$n = 20$	$n = 10$	$n = 20$	$n = 100$	$N = 9 + 1$	teristic Eq.
ω_1	1.00129	1.00122	1.00129	1.00122	1.00117	1.00115	1.00115
ω_2	6.79358	6.50238	6.79358	6.49691	6.27861	6.22835	6.22835
ω_3	9.05139	9.04135	9.05139	9.04118	9.03497	9.03366	9.03366
ω_4	22.12206	21.04961	22.12206	21.03035	20.29573	20.13636	20.13597
ω_5	25.34325	25.21891	25.34325	25.21725	25.16305	25.15322	25.15316
ω_6	46.17493	43.84070	46.17493	43.79855	42.24183	41.93648	41.91899
ω_7	50.46579	49.68282	50.46579	49.67492	49.44604	49.41367	49.40957
ω_8	78.57786	74.79265	78.57786	74.71932	72.06834	72.02385	71.53860

Table 3 Comparison of accuracies of various methods for the beam with a linear spring

	Rayleigh-Ritz: without forced modes		Pseudocharacteristic equation[16]		Rayleigh-Ritz: 1 force-mode,	Exact values from charac-
Frequency	$n = 10$	$n = 20$	$n = 10$	$n = 20$	$N = 9 + 1$	teristic Eq.
ω_1	3.98982	3.98789	3.98982	3.98979	3.98979	3.98979
ω_2	6.21510	6.20625	6.21510	6.20624	6.20499	6.20499
ω_3	15.90992	15.90760	15.90992	15.90760	15.90727	15.90727
ω_4	19.79345	19.69332	19.79345	19.69323	19.67924	19.67922
ω_5	35.56427	35.50733	35.56427	35.50727	35.49878	35.49867
ω_6	39.98816	39.49512	39.98816	39.49471	39.43190	39.43056
ω_7	61.53126	60.53431	61.53126	60.53351	60.42846	60.40820
ω_8	66.14810	65.42392	66.14810	65.42353	65.37827	65.36386

Violation of Complementary Boundary Conditions

Consider now the axial vibration of a uniform bar fixed at one end and free at the other. The boundary-value problem has the form

$$EAu''(x) + \lambda mu(x) = 0, \qquad 0<x<L$$

$$u(0) = 0, \qquad u'(L) = 0 \tag{46}$$

The complementary boundary conditions are: at $x = 0$, $u'(0)$ is unknown and at $x = L$, $u(L)$ is unknown. We select $EA = 1$, $m = 1$, $L = 1$. This problem was analyzed qualitatively in the previous section. A closed-form eigensolution exists in the form $\omega_r = (2r - 1)\pi/2$, $\phi_r(x) = \sin\omega_r x$, $\lambda_r = \omega_r^2$.

Table 4 Frequencies of fixed-free bar by Rayleigh-Ritz method

Frequency	Using polynomials	Fixed-free beam eigenfunctions	Exact values
ω_1	1.570796	1.618747	1.570796
ω_2	4.712389	4.881649	4.712389
ω_3	7.855044	8.081827	7.853982

To solve for the eigensolution using trial functions, we consider two sets of admissible functions $\Psi_{1r}(x)$, $\Psi_{2r}(x)$, such that

$$\Psi_{1r}(x) = x^r \tag{47a}$$

$$\Psi_{2r}(x) = (\sin\beta_r L - \sinh\beta_r L)(\sin\beta_r x - \sinh\beta_r x) + (\cos\beta_r L + \cosh\beta_r L)(\cos\beta_r x - \cosh\beta_r x) \tag{47b}$$

The first set are simple polynomials. The second set are the eigenfunctions of a fixed-free beam in bending, where $\beta_r L$ are the solutions of the associated characteristic equation. Both the displacement and slope are zero at $x = 0$, for all $\Psi_{2r}(x)$, thus violating the complementary boundary condition at $x = 0$. The natural frequencies for $n = 7$ are given in Table 4. As can be seen, the results using the second set of admissible functions are very much in error.

We next examine the error in the eigenfunctions and the residual $R_r(x)$ associated with the approximate eigenfunctions. We define the error between the exact and approximate eigenfunctions by $e_r(x) = \phi_r(x) - \theta_r(x)$ $(r = 1,2,\ldots,n)$. Figures 3-8 plot the error in the eigenfunctions $e_r(x)$, their derivatives $e_r'(x)$, and the residual $R_r(x)$ for the first two eigenfunc-

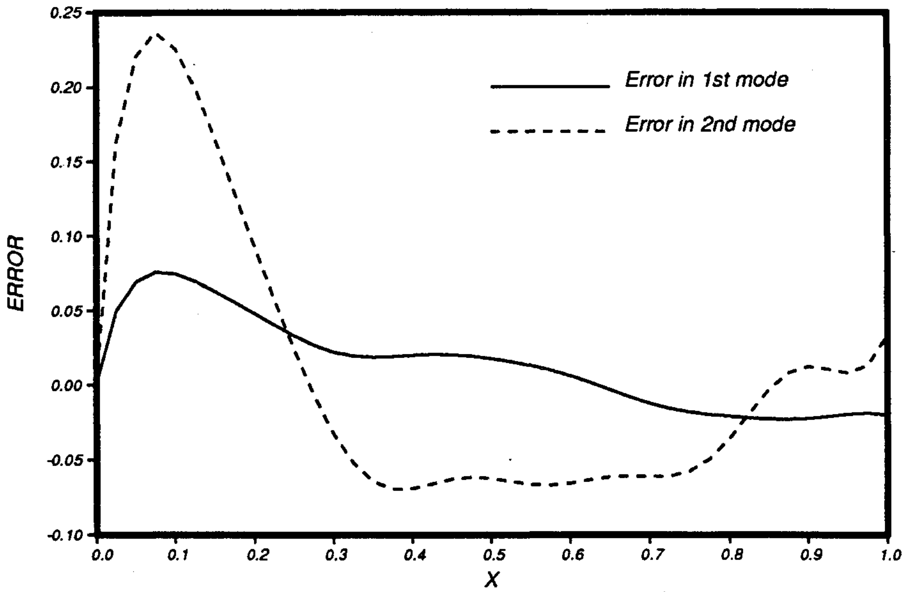

Fig. 3 Error in the first two eigenfunctions (trial functions violate the CBC at $x = 0$).

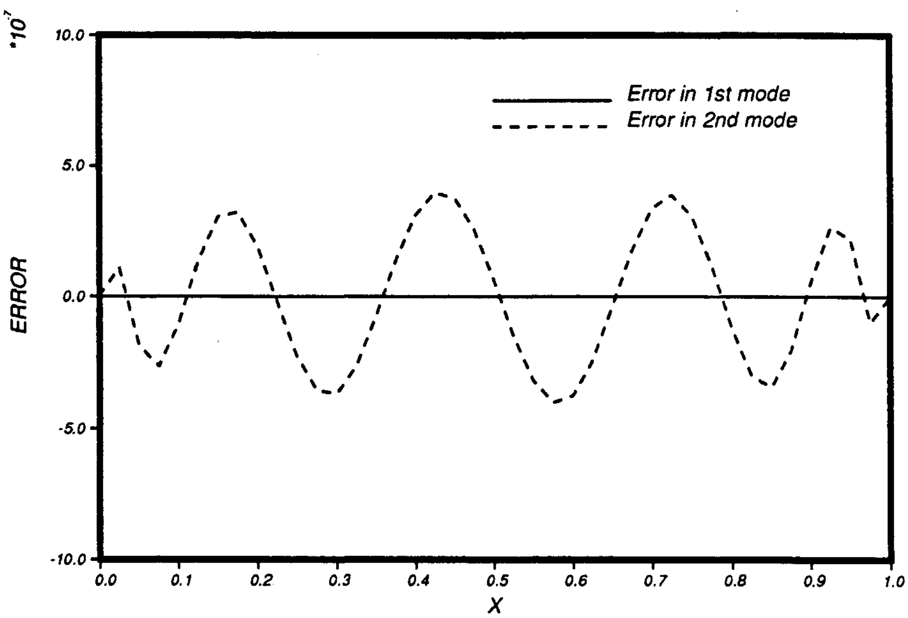

Fig. 4 Error in the first two eigenfunctions (simple polynomials as trial functions).

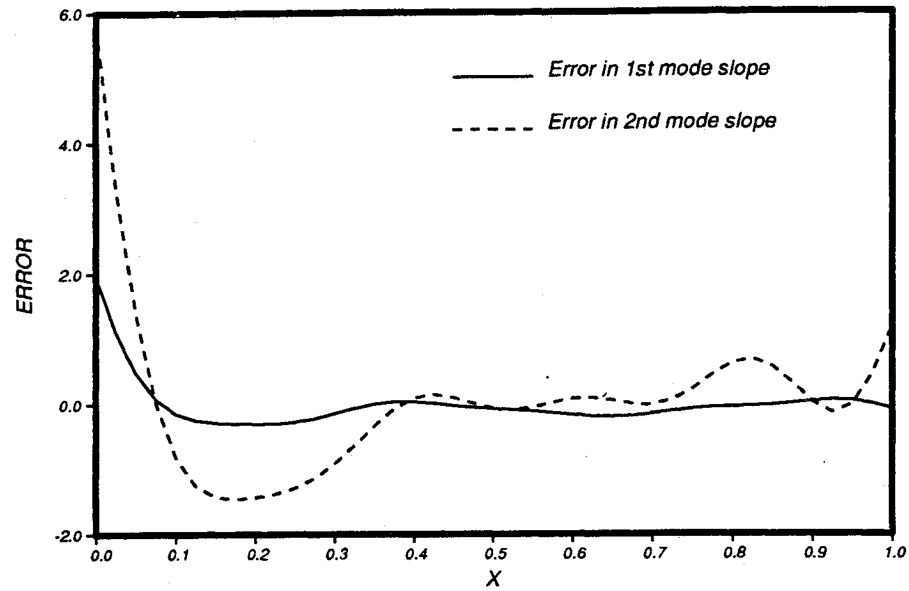

Fig. 5 Error in the slopes of the first two eigenfunctions (trial functions violate the CBC at $x = 0$).

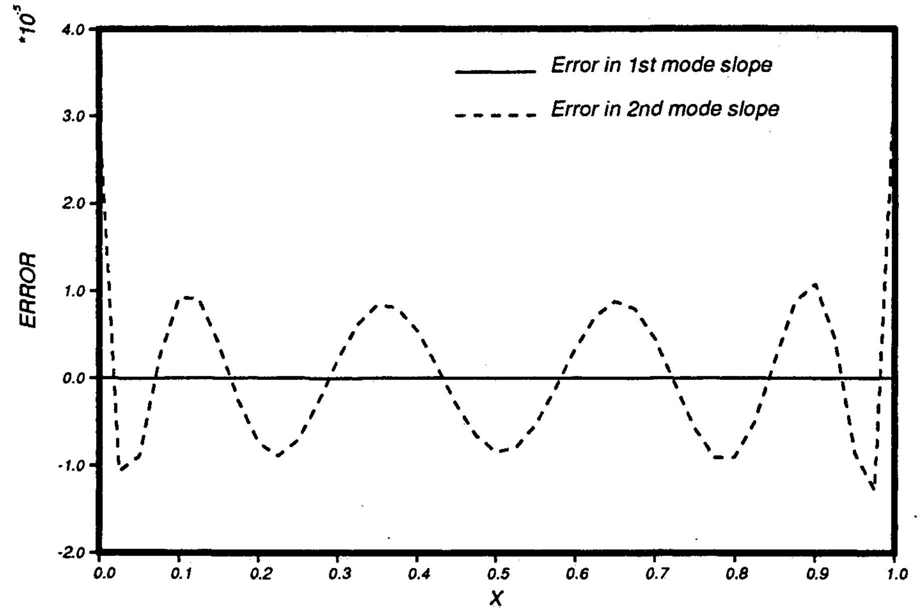

Fig. 6 Error in the slopes of the first two eigenfunctions (simple polynomials as trial functions).

tions for an approximation of order $n = 10$. We observe immediately that all these quantities are much higher for $\Psi_{2r}(x)$, the trial functions that violate the CBC. Moreover, at the point where the CBC is violated ($x = 0$), there is a sudden change in amplitudes, especially for $e_r'(x)$, indicating the Gibbs effect in the first derivative of the solution.

To quantify the amplitudes of the large variations in the vicinity of $x = 0$, we make use of concepts from boundary-layer theory.[18-19] To this end, we subtract Eq. (13) from Eq. (9) and consider the form of the stiffness operator, which gives

$$e_r''(x) - \lambda_r e_r(x) = -R_r(x) - (\Lambda_r - \lambda_r)\theta_r(x)$$

$$r = 1,2,\ldots,n \tag{48}$$

From Figs 5-8, in the neighborhoods of $x = 0$ and $x = 1$ there are boundary layers. We are interested in the boundary layer in the vicinity of $x = 0$. Noting that $e_r(0) = 0$ and $\theta_r(0) = 0$, inside the boundary layer Eq. (48) can be approximated as

$$e_r''(x) \approx -R_r(x), \qquad r = 1,2,\ldots,n \tag{49}$$

We approximate the residual in that neighborhood by $R_r(x) = R_r(0)e^{-x/\epsilon}$, where ϵ is a small parameter. Integration of Eq. (49) yields

$$e_r'(x) \approx c_{or} + \epsilon R_r(0)e^{-x/\epsilon} \tag{50}$$

in which c_{or} is a constant of integration. For the case when the CBC is violated, we have at $x = 0$

$$e_r'(0) \approx c_{or} + \epsilon R_r(0) = \mathcal{O}(1), \qquad r = 1,2,\ldots,n \tag{51}$$

which implies that either c_{or} is of order $\mathcal{O}(1)$ or $R_r(0)$ is of order $\mathcal{O}(1/\epsilon)$. By contrast, when the trial functions do not violate the CBC, $e_r'(0)$ is very small, indicating that $R_r(0)$ is less than order $\mathcal{O}(1/\epsilon)$. We also observe from Figs. 4-6 that for both sets of trial functions amplitudes of the error increase by an order of magnitude as a derivative is taken. For the trial functions that violate the CBC, the error in the vicinity of $x = 0$ is of order $\mathcal{O}(\epsilon)$ for $e_r(x)$, of order $\mathcal{O}(1)$ for $e_r'(x)$, and $\mathcal{O}(1/\epsilon)$ for $e_r''(x)$.

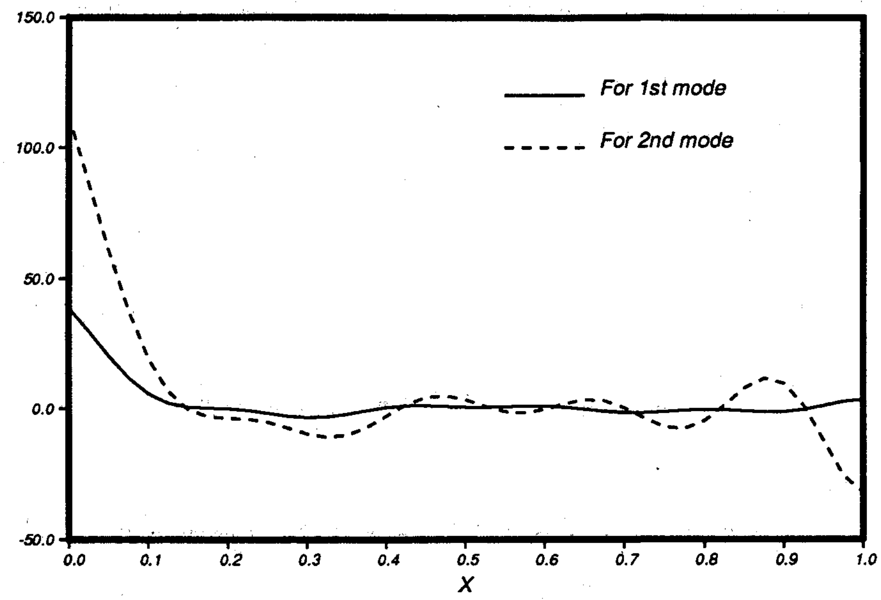

Fig. 7 $R_1(x)$ and $R_2(x)$ for the first two modes (trial functions violate the CBC at $x = 0$).

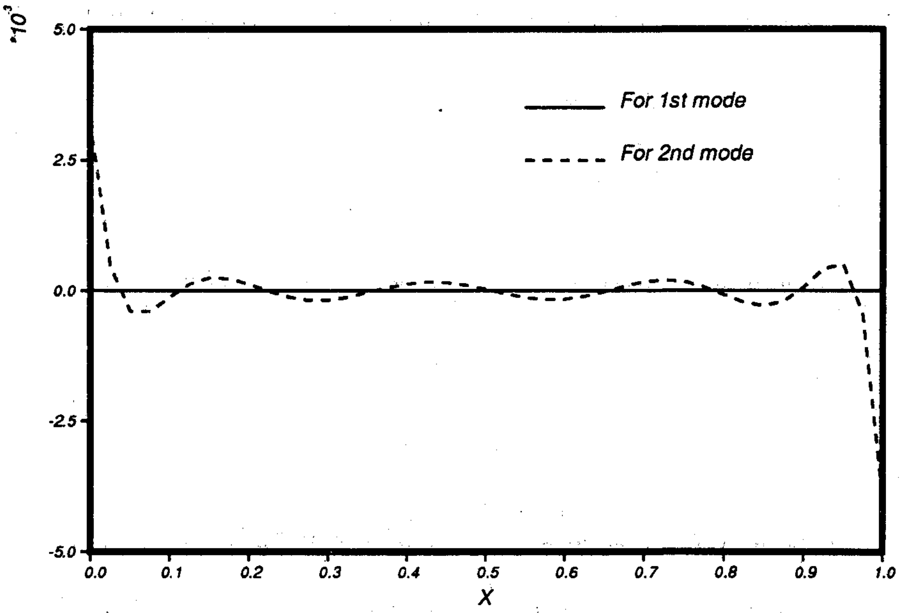

Fig. 8 $R_1(x)$ and $R_2(x)$ for the first two modes (simple polynomials as trial functions).

Substructure Analysis

Consider a uniform beam pinned at both ends. The beam parameters are $EI = 32960$ lb in.^{2}, $m = 2.911*10^{-4}$ lb s^2/in.^{2}, $L = 144$ in. We cut the beam into two at $x = L/2$, model each substructure as a pinned-free beam, and denote the displacement in each substructure by $u_1(x,t)$ $(0<x<L/2)$ and $u_2(x,t)$ $(L/2<x<L)$. The global equations of motion make use of the geometric compatibility constraints $u_1(L/2,t) = u_2(L/2,t)$ and $u_1'(L/2,t) = u_2'(L/2,t)$. Details of the substructure equations can be found in Refs. 8 and 10.

The complementary boundary conditions at the interface $x = L/2$ denote the nonzero shear forces and bending moments

$$u_1''(L/2,t) = u_2''(L/2,t) \neq 0, \qquad u_1'''(L/2,t) = u_2'''(L/2,t) \neq 0 \tag{52}$$

The physical interpretation is that if two bars are constrained to be held together at a point, there must be an internal force and moment holding them together.

We compare the eigenvalues obtained from the substructure analysis. We consider two sets of admissible functions. The first set is simple polynomials $x, x^2, \ldots,$ for $u_1(x)$ and $(L-x), (L-x)^2, \ldots,$ for $u_2(x)$. The second set consists of eigenfunctions of pinned-free and free-pinned beams. We recognize the latter set as trial functions that violate both the complementary boundary conditions given by Eqs. (52).

Table 5 compares the computed frequencies with the exact ones for eight terms in the expansion for each substructure. When the pinned-free modes are used without any constraint

Table 5 Frequencies through substructure synthesis using admissible functions

Frequency	Polynomials as admissible functions	Pinned-free beam eigenfunctions as admissible functions	Exact values
ω_1	5.064625	5.351286	5.064625
ω_2	20.258498	20.264122	20.258498
ω_3	45.581631	48.415582	45.581621
ω_4	81.035472	81.127357	81.033993
ω_5	126.635186	135.255566	126.615610
ω_6	182.370993	182.799473	182.324680

modes added to them, the complementary boundary conditions are violated and the computed frequencies are incorrect.

Table 5 also indicates that the even-numbered frequencies are estimated more accurately than the odd-numbered frequencies when eigenfunctions of a pinned-free beam are used as trial functions. We can explain this by noting that the even-numbered modes have zero second derivatives at $x = L/2$, so that only the CBC associated with the shear balance is violated, which is in the third derivative. The Gibbs effect is in the second derivative for the odd numbered modes and in the third derivative for the even numbered modes, implying faster convergence for the even modes.

Conclusions

Occurrence of the Gibbs phenomenon in approximate solutions to structural mechanics problems by series expansions is investigated. The concepts of complementary boundary condition and interior condition are discussed. The qualitative and quantitative analyses, as well as the numerical examples, clearly show the slow convergence when the trial functions either violate the complementary boundary conditions, or cannot satisfy the interior conditions. When obtaining a series solution to a structural problem, static or dynamic, the trial functions should be selected such that the complementary boundary conditions are not violated and the interior conditions are satisfied as best as possible.

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